Stromberg, page 49.

(a) $f(0) = f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0 = 0 \cdot f(1)$.
$f(2) = f(1+1) = f(1) + f(1) = 2f(1)$.
$f(3) = f(2+1) = f(2) + f(1) = 2f(1) + f(1) = 3f(1)$.
Continuing, by induction, $f(n) = nf(1)\ \forall\, n \in \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$, $0 = f(0) = f(-n+n) = f(-n) + f(n)$, and so $f(-n) = -f(n) = -nf(1)$. Therefore, $f(n) = nf(1)\ \forall\, n \in \mathbb{Z}$.

(b) First note that by induction, for $n \geq 2$, $f(x_1 + \cdots + x_n) = f(x_1) + \cdots + f(x_n)$. Let $\frac{p}{q} \in \mathbb{Q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then

$$pf(1) \underset{(a)}{=} f(p) = f\left(q\left(\frac{p}{q}\right)\right) = f\left(\frac{p}{q} + \cdots + \frac{p}{q}\right) = qf\left(\frac{p}{q}\right),$$

and so $f\left(\frac{p}{q}\right) = \frac{p}{q}f(1)\ \forall\, \frac{p}{q} \in \mathbb{Q}$.

(c) For $x_0$ in $\mathbb{R}$, $f$ is continuous at $x_0 \Leftrightarrow \lim_{x \to x_0} f(x) = f(x_0)$
$\Leftrightarrow \lim_{x \to x_0} (f(x) - f(x_0)) = 0 \Leftrightarrow \lim_{x \to x_0} f(x - x_0) = 0$
$[f(x) = f((x - x_0) + x_0) = f(x - x_0) + f(x_0)]$
$\Leftrightarrow \lim_{y \to 0} f(y) = 0$ [let $y = x - x_0$] $\Leftrightarrow f$ is continuous at 0. Thus, $f$ is continuous at a point $c \in \mathbb{R} \Leftrightarrow f$ is continuous at $0 \Leftrightarrow f$ is continuous on $\mathbb{R}$.

(d) Let $x \in \mathbb{R} \setminus \mathbb{Q}$ and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{Q}$ with $x_n \to x$. Since $f$ is continuous at $x$,

$$f(x) = \lim_{n \to \infty} f(x_n) \underset{(b)}{=} \lim_{n \to \infty} x_n f(1) = xf(1).$$

## 4.4 Consequences of Continuity

1. $f(x) = \dfrac{1}{x}$ on $(0,1)$ works for both (a) and (b).

2. Any constant function or $f(x) = \dfrac{1}{x^2+1}$ maps $(-1,1)$ onto $(\frac{1}{2}, 1]$.

3. By Proposition 4.8 and the Completeness Axiom, $\beta = \sup\{f(x) : x \in [a,b]\} \in \mathbb{R}$. We want to find an $M$ in $[a,b]$ such that $f(M) = \beta$. By

Proposition 2.5, $\forall\, n \in \mathbb{N}\ \exists\, x_n \in [a,b]$ such that $\beta - \frac{1}{n} < f(x_n) \le \beta$. By Theorems 3.10 and 3.3, $\exists$ a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n\in\mathbb{N}}$ and an $M \in [a,b]$ such that $x_{n_k} \underset{k}{\to} M$. Since $f$ is continuous at $M$, $f(x_{n_k}) \underset{k}{\to} f(M)$; and since $\beta - \frac{1}{n} < f(x_n) \le \beta\ \forall\, n \in \mathbb{N}$, $f(x_{n_k}) \underset{k}{\to} \beta$. By the uniqueness of limits, $f(M) = \beta$.

4. $f(0) > 0, f(1) < 0, f(2) > 0$. By the Intermediate Value Theorem (IVT) or Corollary 4.2, $f$ has a root between 0 and 1 and a root between 1 and 2.

5. Let $f(x) = a_nx^n + \cdots + a_1x + a_0$ where $n$ is odd, the $a_i$'s $\in \mathbb{R}$, and $a_n \neq 0$.
<u>Case 1</u>: $a_n > 0$. By Example 4.15, $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. Thus, $\exists\, a < 0 < b$ with $f(a) < 0 < f(b)$. By the IVT $\exists\, c \in (a,b)$ with $f(c) = 0$.
<u>Case 2</u>: $a_n < 0$. By Example 4.15, $f(x) \to -\infty$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to -\infty$. Thus, $\exists\, a < 0 < b$ with $f(a) > 0 > f(b)$. By the IVT $\exists\, c \in (a,b)$ with $f(c) = 0$.

6. Let $f(x) = a_nx^n + \cdots + a_1x + a_0$ where $n$ is even and the $a_i$'s $\in \mathbb{R}$. If $n = 0$, then $f(x) = a_0$ has an absolute maximum and an absolute minimum at every point of $\mathbb{R}$. Let $n \ge 2$ with $a_n \neq 0$.
<u>Case 1</u>: $a_n > 0$. Since $f(x) \to \infty$ as $x \to \pm\infty$ by Example 4.15, $\exists \alpha > 0$ with $\alpha$ in the range of $f$. So $\exists\, a < 0 < b$ with $f(x) > \alpha\ \forall x \in \mathbb{R} \setminus [a,b]$. By Theorem 4.2, $f$ has an absolute minimum on $[a,b]$ and this is the absolute minimum of $f$ on $\mathbb{R}$. (If $m \in [a,b]$ with $f(m)$ the absolute minimum of $f$ on $[a,b]$, then $f(m) \le \alpha$.)
<u>Case 2</u>: $a_n < 0$. Since $f(x) \to -\infty$ as $x \to \pm\infty$ by Example 4.15, $\exists \beta < 0$ with $\beta$ in the range of $f$. So $\exists\, a < 0 < b$ with $f(x) < \beta$ $\forall x \in \mathbb{R} \setminus [a,b]$. By Theorem 4.2, $f$ has an absolute maximum on $[a,b]$ and this is the absolute maximum of $f$ on $\mathbb{R}$.

7. Suppose $f : (a,b) \to \mathbb{Z}$ is nonconstant. Then $\exists\, n_1$ and $n_2 \in \mathbb{Z}$, $n_1 \neq n_2$, with both $n_1$ and $n_2$ in the range of $f$. We may assume $n_1 < n_2$. Since $f$ is continuous on $(a,b)$, by the IVT, the interval $[n_1, n_2]$ is contained in the range of $f$, a contradiction to $f$ being integer valued. Therefore, $f$ is a constant function.

8. Define $g : \left[0, \frac{1}{2}\right] \to \mathbb{R}$ by $g(x) = f(x) - f\left(x + \frac{1}{2}\right)$. Then $g$ is continuous on $\left[0, \frac{1}{2}\right]$, $g(0) = f(0) - f\left(\frac{1}{2}\right)$, and $g\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) - f(1) = f\left(\frac{1}{2}\right) - f(0) = -g(0)$. If $g(0) = 0$, then $f(0) = f\left(\frac{1}{2}\right)$. If $g(0) \neq 0$, the IVT implies that $\exists\ c \in \left(0, \frac{1}{2}\right)$ such that $g(c) = 0$. Hence, $f(c) = f\left(c + \frac{1}{2}\right)$.

9. Let $f(x) = x - \cos x$ for $x$ in $\left[0, \frac{\pi}{2}\right]$. Then $f$ is continuous on $\left[0, \frac{\pi}{2}\right]$, $f(0) = -1$, and $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$. By the IVT, $\exists\, c \in \left(0, \frac{\pi}{2}\right)$ with $f(c) = 0$. Hence, $c = \cos c$.
At this point the question about $x = \sin x$ is meant to give the students something to think about. Ask your students to graph $y = x$ and $y = \sin x$ (which intersect only at $x = 0$) on the same axes. That $\sin x < x\ \forall\ x > 0$ is Example 5.3 and that $\sin x > x\ \forall\ x < 0$ is Exercise 5.2.10 in the next chapter. The first part is a Calculus I type argument using facts about the derivative.

10. Let $f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0\,. \end{cases}$ Then $f$ is discontinuous at $0$ by Example 4.6, but $f$ satisfies the intermediate value property. (If $-1 \leq \alpha \leq 1$, choose $t > 0$ such that $\sin t = \alpha$. Letting $x = \frac{1}{t}$, $\sin\frac{1}{x} = \alpha$.)
More simply, $g(x) = \begin{cases} x+1 & \text{if } x \leq 0 \\ x-1 & \text{if } x > 0 \end{cases}$ also serves as an example.

## 4.5 Uniform Continuity

1. Let $\varepsilon = 1$ and let $\delta > 0$. To show that $f$ is not uniformly continuous on $\mathbb{R}$, we need to find $x$ and $y$ in $\mathbb{R}$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq 1$. First let $x > 0$ and $y = x + \frac{\delta}{2}$. Then $|x - y| = \frac{\delta}{2} < \delta$ and

$$\begin{aligned} |f(x) - f(y)| &= |x^3 - y^3| \\ &= |x - y|\,|x^2 + xy + y^2| \\ &= \frac{\delta}{2}\left[x^2 + x\left(x + \frac{\delta}{2}\right) + \left(x + \frac{\delta}{2}\right)^2\right] \end{aligned}$$

$$\geq \frac{\delta}{2}(3x^2).$$

Now we fix $x$. Let $x = \sqrt{\frac{2}{\delta}}$. Then $|f(x) - f(y)| \geq \frac{\delta}{2}\left(\frac{3(2)}{\delta}\right) = 3 > 1$.

2. Let $x_n = 2 + \frac{1}{n}$ $\forall n \in \mathbb{N}$. Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(2, \infty)$ but $(f(x_n))_{n\in\mathbb{N}}$ is not Cauchy. By Theorem 4.5, $f$ is not uniformly continuous on $(2, \infty)$.

3. $\left(\dfrac{1}{\frac{\pi}{2} + n\pi}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\left(0, \frac{\pi}{2}\right]$ but its image under $f$, $((-1)^n)_{n\in\mathbb{N}}$, is not Cauchy. Hence, Theorem 4.5 $\Rightarrow$ $f$ is not uniformly continuous on $\left(0, \frac{\pi}{2}\right]$. This also follows from Theorem 4.7 since $\sin\frac{1}{x}$ cannot be continuously extended to 0 by Example 4.6.

4. <u>Case 1</u>: $m = 0$. So $f(x) = b$. Let $\varepsilon > 0$, and let $\delta > 0$ be arbitrary. For $x$ and $y$ in $\mathbb{R}$ with $|x - y| < \delta$, $|f(x) - f(y)| = 0 < \varepsilon$.
   <u>Case 2</u>: $m \neq 0$. Let $\varepsilon > 0$ and let $\delta = \frac{\varepsilon}{|m|}$. For $x$ and $y$ in $\mathbb{R}$ with $|x - y| < \delta$,

$$\begin{aligned} |f(x) - f(y)| &= |mx + b - (my + b)| \\ &= |m|\,|x - y| \\ &< |m|\,\delta \\ &= |m|\,\frac{\varepsilon}{|m|} = \varepsilon. \end{aligned}$$

   By Definition 4.6, $f$ is uniformly continuous on $\mathbb{R}$.

5. To show $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[1, \infty)$, let $\varepsilon > 0$ and let $\delta = \frac{\varepsilon}{2}$. For $x$ and $y$ in $[1, \infty)$ with $|x - y| < \delta$,

$$\begin{aligned} |f(x) - f(y)| &= \left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \frac{|x^2 - y^2|}{x^2y^2} \\ &= \frac{|x - y|\,(x + y)}{x^2y^2} \\ &= |x - y|\left(\frac{1}{xy^2} + \frac{1}{x^2y}\right) \end{aligned}$$

$$\begin{aligned} &\le |x-y|(1+1) \qquad (xy^2 \ge 1 \text{ and } x^2y \ge 1) \\ &= 2|x-y| < 2\delta = \varepsilon. \end{aligned}$$

To show $f(x) = \dfrac{1}{x^2}$ is not uniformly continuous on $(0,1]$, note that $\left(\dfrac{1}{n}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(0,1]$ but $\left(f\left(\dfrac{1}{n}\right)\right)_{n\in\mathbb{N}} = (n^2)_{n\in\mathbb{N}}$ is not Cauchy. By Theorem 4.5, $f$ is not uniformly continuous on $(0,1]$.

6. Let $\varepsilon > 0$. Since $f$ is uniformly continuous on $[a,b]$, $\exists\ \delta_1 > 0$ such that $x,y \in [a,b]$ with $|x-y| < \delta_1 \Rightarrow |f(x)-f(y)| < \dfrac{\varepsilon}{2}$. Since $f$ is uniformly continuous on $D$, $\exists\ \delta_2 > 0$ such that $x,y \in D$ with $|x-y| < \delta_2 \Rightarrow |f(x)-f(y)| < \dfrac{\varepsilon}{2}$. Let $\delta = \min\{\delta_1, \delta_2\}$. Let $x,y \in [a,b] \cup D$ with $|x-y| < \delta$. If both $x,y \in [a,b]$ or both $x,y \in D$, then $|f(x)-f(y)| < \varepsilon/2$. Suppose $x \in [a,b]$ and $y \in D$. Since $x \le b \le y$, $b - x < \delta$ and $y - b < \delta$. Therefore,

$$|f(x)-f(y)| \le |f(x)-f(b)| + |f(b)-f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

7. Let $\varepsilon > 0$ and let $\delta = 2\varepsilon$. For $x,y \in [1,\infty)$ with $|x-y| < \delta$,

$$\begin{aligned} |f(x)-f(y)| &= |\sqrt{x}-\sqrt{y}| \cdot \left|\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right| \\ &= \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \\ &\le \frac{1}{2}|x-y| \qquad (\text{since } \sqrt{x}+\sqrt{y} \ge 1+1 = 2) \\ &< \frac{\delta}{2} = \varepsilon. \end{aligned}$$

Therefore, $f$ is uniformly continuous on $[1,\infty)$. By Theorem 4.4, $f$ is uniformly continuous on $[0,1]$. By Exercise 6, $f$ is uniformly continuous on $[0,\infty)$. [An $\varepsilon-\delta$ argument to show that $f(x) = \sqrt{x}$ is uniformly continuous on $[0,1]$ is given at the end of Chapter 8 in this manual.]

8. Suppose $f$ is not bounded on $D$. Then $\forall\ n \in \mathbb{N}$, $\exists\ x_n \in D$ with $|f(x_n)| > n$. Since $D$ is bounded, by Theorem 3.10 the sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent and hence Cauchy subsequence $(x_{n_k})_{k=1}^{\infty}$. Since $|f(x_{n_k})| > n_k \ge k\ \forall\ k \in \mathbb{N}$, $(f(x_{n_k}))_{k=1}^{\infty}$ is unbounded and hence not Cauchy. By Theorem 4.5, $f$ is not uniformly continuous on $D$.

9. Let $\varepsilon > 0$. Since $f$ is uniformly continuous on $D$, $\exists\ \delta_1 > 0$ such that $x, y \in D$ with $|x-y| < \delta_1 \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}$. Since $g$ is uniformly continuous on $D$, $\exists\ \delta_2 > 0$ such that $x, y \in D$ with $|x-y| < \delta_2 \Rightarrow |g(x) - g(y)| < \frac{\varepsilon}{2}$. Let $\delta = \min\{\delta_1, \delta_2\}$. For $x, y \in D$ with $|x-y| < \delta$,
$$\begin{aligned} |(f+g)(x) - (f+g)(y)| &= |f(x) + g(x) - (f(y) + g(y))| \\ &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$
Therefore, $f+g$ is uniformly continuous on $D$.
By Exercise 4, $f(x) = x$ is uniformly continuous on $\mathbb{R}$. Let $g = f$. Then $f \cdot g(x) = x^2$ is not uniformly continuous on $\mathbb{R}$ by Example 4.19.

10. We have that $f$ is uniformly continuous on $(a, b)$. To show that we can continuously extend $f$ to $b$, by the Remark following Example 4.14, it suffices to show that $\lim\limits_{x \to b} f(x)$ is a real number. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $(a, b)$ with $x_n \to b$. By Theorem 4.5, $(f(x_n))_{n \in \mathbb{N}}$ is a Cacuhy sequence and so $\exists\ L \in \mathbb{R}$ such that $f(x_n) \to L$. We claim that $\lim\limits_{x \to b} f(x) = L$. Let $(y_n)_{n \in \mathbb{N}}$ be a sequence in $(a, b)$ with $y_n \to b$. As above, $\exists\ L_1 \in \mathbb{R}$ such that $f(x_n) \to L_1$. By Proposition 4.6, we must show that $L = L_1$.
Consider the sequence $(z_n)_{n \in \mathbb{N}} = (x_1, y_1, x_2, y_2, x_3, y_3, \ldots)$ in $(a, b)$. Since $z_n \to b$ by Exercise 3.3.4, $\exists\ M \in \mathbb{R}$ such that $f(z_n) \to M$. Since $(f(x_n))_{n \in \mathbb{N}}$ and $(f(y_n))_{n \in \mathbb{N}}$ are subsequences of $(f(z_n))_{n \in \mathbb{N}}$, $f(x_n) \to M$ and $f(y_n) \to M$. By the uniqueness of limits, $L = M = L_1$.

11. Let $f(x) = \begin{cases} -1 & \text{if } x < \sqrt{2} \\ 1 & \text{if } x > \sqrt{2}. \end{cases}$ Since $\lim\limits_{x \to \sqrt{2}} f(x)$ does not exist, $f$ cannot be continuously extended to $\mathbb{R}$. However, $f$ is continuous on $\mathbb{R} \setminus \{\sqrt{2}\}$, and so $f$ is continuous on the subset $\mathbb{Q}$.

12. To show $g$ is well-defined, in the notation of the hint, note that Theorem 4.5 $\Rightarrow (f(x_n))_{n \in \mathbb{N}}$ is Cauchy and so $(f(x_n))_{n \in \mathbb{N}}$ converges to some $\alpha \in \mathbb{R}$. Similarly, $(f(y_n))_{n \in \mathbb{N}}$ converges to some $\beta \in \mathbb{R}$. We must show that $\alpha = \beta$.
Let $\varepsilon > 0$. Since $f$ is uniformly continuous on $\mathbb{Q}$, $\exists\ \delta > 0$ such that $p, q \in \mathbb{Q}$ with $|p - q| < \delta \Rightarrow |f(p) - f(q)| < \frac{\varepsilon}{3}$. Since $x_n \to x$ and $y_n \to x$, $\exists\ n_1, n_2 \in \mathbb{N}$ such that $n \geq n_1 \Rightarrow |x_n - x| < \frac{\delta}{2}$ and $n \geq n_2 \Rightarrow$

$|y_n - x| < \dfrac{\delta}{2}$. So $n \geq \max\{n_1, n_2\} \Rightarrow |x_n - y_n| \leq |x_n - x| + |x - y_n| < \delta$. Therefore, $n \geq \max\{n_1, n_2\} \Rightarrow |f(x_n) - f(y_n)| < \dfrac{\varepsilon}{3}$. Since $f(x_n) \to \alpha$ and $f(y_n) \to \beta$, $\exists\, n_3, n_4 \in \mathbb{N}$ such that $n \geq n_3 \Rightarrow |f(x_n) - \alpha| < \dfrac{\varepsilon}{3}$ and $n \geq n_4 \Rightarrow |f(y_n) - \beta| < \dfrac{\varepsilon}{3}$. Fix an $n \geq \max\{n_1, n_2, n_3, n_4\}$. Then

$$\begin{aligned} |\alpha - \beta| &\leq |\alpha - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - \beta| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $\alpha = \beta$. Therefore, $g$ is well-defined, and clearly $g|_{\mathbb{Q}} = f$.

<u>To show</u>: $g$ is uniformly continuous on $\mathbb{R}$. Let $\varepsilon > 0$. Since $f$ is uniformly continuous on $\mathbb{Q}$ and $g = f$ on $\mathbb{Q}$, $\exists\, \delta_1 > 0$ such that $p, q \in \mathbb{Q}$ with $|p - q| < \delta_1 \Rightarrow |g(p) - g(q)| < \dfrac{\varepsilon}{3}$. Let $\delta = \dfrac{\delta_1}{3}$ and let $x, y \in \mathbb{R}$ with $|x - y| < \delta$. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{Q}$ with $x_n \to x$ and $y_n \to y$. Choose $n_1 \in \mathbb{N}$ such that $n \geq n_1 \Rightarrow |x_n - x| < \delta$ and $|y_n - y| < \delta$. Then $n \geq n_1 \Rightarrow |x_n - y_n| \leq |x_n - x| + |x - y| + |y - y_n| < 3\delta = \delta_1$. By definition of $g$ and since $g|_{\mathbb{Q}} = f$, $g(x_n) \to g(x)$ and $g(y_n) \to g(y)$. Hence, $\exists\, n_2, n_3 \in \mathbb{N}$ such that $n \geq n_2 \Rightarrow |g(x_n) - g(x)| < \dfrac{\varepsilon}{3}$ and $n \geq n_3 \Rightarrow |g(y_n) - g(y)| < \dfrac{\varepsilon}{3}$. Fix an $n \geq \max\{n_1, n_2, n_3\}$. Then

$$\begin{aligned} |g(x) - g(y)| &\leq |g(x) - g(x_n)| + |g(x_n) - g(y_n)| + |g(y_n) - g(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

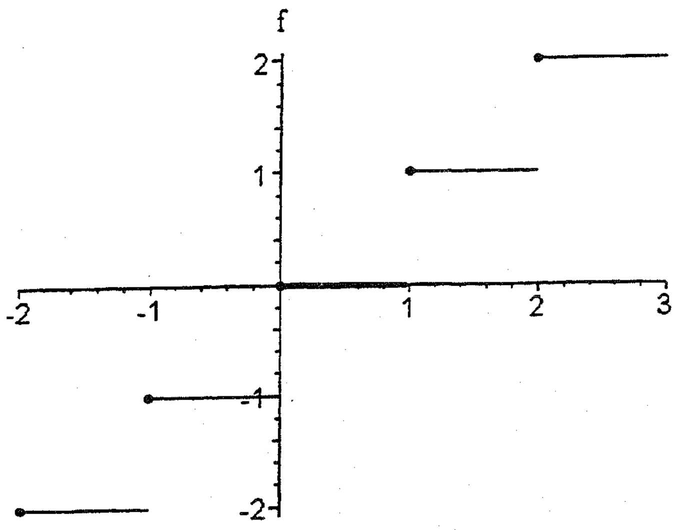
and so $g$ is uniformly continuous on $\mathbb{R}$.

<u>To show</u>: $g$ is unique. ($g$ is unique since two continuous functions which agree on a dense set agree everywhere.) Suppose $h$ is a continuous extension of $f$ to $\mathbb{R}$. Then $h = g$ on $\mathbb{Q}$. Let $x \in \mathbb{R} \setminus \mathbb{Q}$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{Q}$ with $x_n \to x$. Then

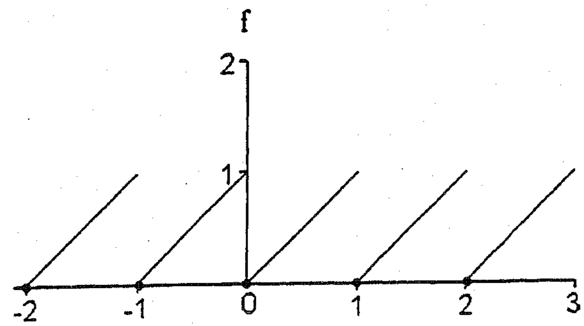
$$h(x) = \lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} g(x_n) = g(x).$$

## 4.6 Discontinuities and Monotone Functions

1. $f$ has a discontinuity of the first kind at each integer, and $f$ is monotone increasing. (Note that $f(n+) = n$ and $f(n-) = n - 1\ \forall\, n \in \mathbb{Z}$.)




2. $f$ has a discontinuity of the first kind at each integer, and $f$ is not monotone. (Note that $f(n+) = 0$ and $f(n-) = 1 \; \forall \, n \in \mathbb{Z}$.)




3. Let $c \in \mathbb{R}$. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{Q} \cap (-\infty, c)$ with $x_n \to c$, and let $(y_n)_{n \in \mathbb{N}}$ be a sequence in $(\mathbb{R} \setminus \mathbb{Q}) \cap (-\infty, c)$ with $y_n \to c$. Since $f(x_n) \to 1$ and $f(y_n) \to 0$, $f(c-)$ does not exist. Similarly, let $(s_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{Q} \cap (c, \infty)$ with $s_n \to c$, and let $(t_n)_{n \in \mathbb{N}}$ be a sequence in $(\mathbb{R} \setminus \mathbb{Q}) \cap (c, \infty)$ with $t_n \to c$. Since $f(s_n) \to 1$ and $f(t_n) \to 0$, $f(c+)$ does not exist.

4. From Exercise 4.2.3, $f$ is discontinuous at every $c \neq 0$. Let $c \neq 0$, let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{Q} \setminus \{c\}$ with $x_n \to c$, and let $(y_n)_{n \in \mathbb{N}}$ be

a sequence in $(\mathbb{R} \setminus \mathbb{Q}) \setminus \{c\}$ with $y_n \to c$. Then $f(x_n) = x_n \to c$ and $f(y_n) \to 0$. Since $c \neq 0$, by restricting the above sequences to each side of $c$, $f(c-)$ and $f(c+)$ do not exist. Therefore, $f$ has a discontinuity of the second kind at every $c \neq 0$.

5. Let $x < y$ in $I$. Then $f$ is monotone increasing on $I \Leftrightarrow f(x) \leq f(y) \Leftrightarrow -f(x) \geq -f(y) \Leftrightarrow -f$ is monotone decreasing on $I$. (This observation will allow us to use the results in the text for monotone increasing functions to obtain the corresponding results for monotone decreasing functions.)

6. Let $f$ be monotone decreasing on $I$ and let $c \in I$. By the previous exercise, $-f$ is monotone increasing on $I$.
   Case 1: $c$ an interior point of $I$. From the proof of Theorem 4.8,

$$\begin{aligned} \sup_{x \in I \cap (-\infty, c)} (-f(x)) &= -f(c-) \leq -f(c) \\ &\leq -f(c+) = \inf_{x \in I \cap (c,\infty)} (-f(x)). \end{aligned} \tag{2}$$

   By Exercise 2.2.4 with $b = -1$, $\sup_{x \in I \cap (-\infty,c)} (-f(x)) = -\inf_{x \in I \cap (-\infty,c)} f(x)$ and $\inf_{x \in I \cap (c,\infty)} (-f(x)) = -\sup_{x \in I \cap (c,\infty)} f(x)$. Making these replacements in (2) and multiplying (2) by $-1$, we obtain

$$\inf_{x \in I \cap (-\infty,c)} f(x) = f(c-) \geq f(c) \geq f(c+) = \sup_{x \in I \cap (c,\infty)} f(x).$$

   Case 2: $c$ a right endpoint of $I$. From the proof of Theorem 4.8 and Exercise 2.2.4,

$$f(c-) = -[-f(c-)] = -\sup_{x \in I \cap (-\infty,c)} (-f(x)) = \inf_{x \in I \cap (-\infty,c)} f(x).$$

   Case 3: $c$ is a left endpoint of $I$. As in Case 2,

$$f(c+) = -[-f(c+)] = -\inf_{x \in I \cap (c,\infty)} (-f(x)) = \sup_{x \in I \cap (c,\infty)} f(x).$$

7. Let $f$ be monotone decreasing on $I$, and let $c, d \in I$ with $c < d$. Since $-f$ is monotone increasing on $I$, from the proof of Lemma 4.3 we have $-f(c+) \leq -f(d-)$. Multiplying by $-1$ we obtain $f(c+) \geq f(d-)$.

8. Let $f$ be monotone decreasing on $I$. Since $-f$ is monotone increasing on $I$, the proof of Theorem 4.9 implies that the set of discontinuities of $-f$ is countable. Since $f$ and $-f$ have the same discontinuities, the set of discontinuities of $f$ is countable.

9. Let $f(x) = \tan x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $f\left(-\frac{\pi}{2}+\right) = -\infty$ and $f\left(\frac{\pi}{2}-\right) = +\infty$. The difference between this and Theorem 4.8 is that $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ are not in the domain of $f$.

10. If $f$ is discontinuous at $c \in I$, then no number strictly between $f(c-)$ and $f(c+)$, except possibly $f(c)$, can be in the range of $f$ since $f$ is monotone. Therefore, $f$ could not satisfy the intermediate value property.

11. (a) Note that either $f(x) = 0$ or $f(x)$ is a sum of positive terms. Therefore, $f(x) \geq 0 \ \forall\, x \in \mathbb{R}$. Let $x < y$. Since $\{n : x_n < x\} \subset \{n : x_n < y\}$, $f(x) \leq f(y)$.

    (b) Fix $k \in \mathbb{N}$. Since $f$ is monotone increasing on $\mathbb{R}$,

$$f(x_k-) = \sup_{x<x_k} f(x) = \sup_{x<x_k} \left( \sum_{\{n:x_n<x\}} \left(\frac{1}{2}\right)^n \right) = \sum_{\{n:x_n<x_k\}} \left(\frac{1}{2}\right)^n,$$

    and

$$f(x_k+) = \inf_{x>x_k} f(x) = \inf_{x>x_k} \left( \sum_{\{n:x_n<x\}} \left(\frac{1}{2}\right)^n \right) = \sum_{\{n:x_n\leq x_k\}} \left(\frac{1}{2}\right)^n.$$

    (For the last equality, if $m \in \mathbb{N}$ is such that $x_k < x_m$, then for any $x$ with $x_k < x < x_m$, $m$ is not included in the set of indicies $\{n : x_n < x\}$.) Therefore, $f(x_k+) - f(x_k-) = \left(\frac{1}{2}\right)^k > 0$, and so $f$ is discontinuous at $x_k$.

    (c) Let $c \in \mathbb{R} \setminus E$. Replacing $x_k$ by $c$ in part (b), we have

$$f(c-) = \sum_{\{n:x_n<c\}} \left(\frac{1}{2}\right)^n = \sum_{\{n:x_n\leq c\}} \left(\frac{1}{2}\right)^n = f(c+).$$

    Since $f$ is monotone increasing, $f(c-) = f(c) = f(c+)$, and so $f$ is continuous at $c$.